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REAL BIFURCATIONS OF TWO-UNIT SYSTEMS WITH ROLLING†

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A geometrical interpretation is proposed of the stability conditions for steady solutions of dynamical systems with simple symmetry in the Lyapunov-critical case, i.e. when the matrix of the linearization has one zero eigenvalue and all other eigenvalues have negative real parts. The change in the nature of the stability of a singular point when the parameter is varied is associated with bifurcations, represented by cusp and butterfly singularities of the manifolds of steady states. Analytic and numerical constructions are given of the bifurcation sets of the two-parameter families of steady states of two-unit systems with rolling, and the relationship of the system parameters responsible for the unsafe-safe boundary of the stability domain is determined.

1. INTRODUCTION

Using the phenomenological approach [1], two-unit systems with rolling [2] may be classified, under certain assumptions, as dynamical systems with very simple symmetry, defined by stipulating that their right-hand sides are odd functions of the state variable x

$$\dot{x} = f(x,\upsilon), \quad f(-x,\upsilon) = -f(x,\upsilon) \tag{1.1}$$

$$x, f \in R^n, \quad \upsilon \in R_+^1$$

In such systems the point $x = 0$ is necessarily singular (a symmetric solution [3]). Consider an n-dimensional system whose linearization matrix has one eigenvalue equal to zero, while the remaining eigenvalues have negative real parts. According to the Lyapunov reduction principle [4, 5], the problem of whether the solution $x = 0$ of this system is stable may be solved by transforming to a one-dimensional dynamical system for the critical variable. In this paper the solution of the stability problem is reduced to analysing two finite equations obtained by methods analogous to the elimination of the non-critical variables from the critical equation in Lyapunov's theory.

The equations of perturbed motion of sufficiently smooth dynamical systems (1.1) in the neighbourhood of the state $x = 0$ involve only odd powers of the variables

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(\upsilon)x_j + \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} a_{klm}^{(i)}(\upsilon)x_k x_l x_m + \tag{1.2}$$

$$+o\left(|x|^3\right), \quad a_{ij} = \text{const}, \quad a_{klm}^{(i)} = a_{kml}^{(i)} = a_{mkl}^{(i)} = \text{const} \quad (i = 1,\dots,n)$$

We will concentrate on the case $n = 2$. The special features of the reduction of the case $n > 2$ to the two-dimensional case will be illustrated below in the problem of two-unit systems with rolling. The characteristic equation of the system

$$\dot{x}_i = f_i(x_1, x_2), \quad f_i(x_1, x_2) \equiv a_{i1}x_1 + \tag{1.3}$$

$$+a_{i2}x_2 + a_{111}^{(i)}x_1^3 + 3a_{112}^{(i)}x_1^2x_2 + 3a_{122}^{(i)}x_1x_2^2 + a_{222}^{(i)}x_2^3 + \dots \quad (i = 1,2)$$

is

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$$\lambda^2 + p\lambda + q = 0, \quad p = -\left[\operatorname{div}(f_1, f_2)\right]_{x_1=0, x_2=0} = \tag{1.4}$$
$$= -(a_{11} + a_{22}), \quad q = \left[D(f_1, f_2) / D(x_1, x_2)\right]_{(0,0)} = a_{11}a_{22} - a_{12}a_{21}$$

Let (0, 0) be a simple singular point for a non-critical value of the characteristic parameter υ, and let us assume that $p > 0$; all the coefficients in (1.3) are continuous functions of the parameter; at subcritical values of the parameter the Poincaré index of the origin is 1 and at supercritical values it is –1. This situation frequently occurs in problems of the dynamics of simple and multiple-unit systems with rolling [2, 6]. At a critical value of the parameter ($q = 0, \lambda_2 = -p$), the problem of stability is solved by the sign of the Lyapunov coefficient [7]

$$g = \left(a_{11}^2 + a_{12}a_{21}\right)^{-3}\Delta, \quad \Delta = a_{11}^3\left(a_{11}a_{222}^{(2)} - a_{21}a_{222}^{(1)}\right) + \tag{1.5}$$
$$+3a_{11}^2 a_{12}\left(a_{21}a_{122}^{(1)} - a_{11}a_{122}^{(2)}\right) + 3a_{11}a_{12}^2(a_{11}a_{112}^{(2)} -$$
$$-a_{21}a_{112}^{(1)}\Big) + a_{12}^3\left(a_{21}a_{111}^{(1)} - a_{11}a_{111}^{(2)}\right)$$

When $g < 0$, the solution $x_1 = 0, x_2 = 0$ of system (1.3) is asymptotically stable, when $g > 0$, it is unstable, while when $g = 0$, one must include fifth-order terms.

2. A GEOMETRIC INTERPRETATION OF THE STABILITY CONDITIONS FOR $g \neq 0$

Under the assumptions of the implicit function theorem, the equations

$$f_i(x_1, x_2) = 0 \quad (i = 1, 2) \tag{2.1}$$

define certain curves in the x_1x_2 plane in the neighbourhood of (0, 0)

$$x_2 = F_i(x_1) = F'_{i0}x_1 + \frac{1}{3!}F'''_{i0}x_1^3 + \ldots \quad \left(F'_{i0} = F'_i(0), \ F'''_{i0} = F'''_i\ (0)\right) \tag{2.2}$$

whose slopes are, by (1.3)

$$\gamma_i = F'_{i0}, \quad \gamma_1 = -a_{11} / a_{12}, \quad \gamma_2 = -a_{21} / a_{22} \tag{2.3}$$

In the critical case $\gamma_1 = \gamma_2$, i.e. the curves (2.2) touch at the origin. If $F'''_{10} \neq F'''_{20}$, they have a threefold intersection of three-point contact at the origin [8]. Let γ^*_i denote the value of γ_i in subcritical position and let us introduce a new quantity $g_1 = (\gamma_1 - \gamma_2)^*\ (F'''_{20} - F'''_{10})$. Analysis shows that in the critical case (in Lyapunov's sense), when the characteristic equation has one zero root, a *threefold singular point is generated* at the origin of the x_1x_2 plane if the differences

$$(\gamma_1 - \gamma_2)^* \text{ and } F'''_{10} - F'''_{20} \tag{2.4}$$

have the same signs. Thus, the condition for a generation bifurcation is $g_1 < 0$ (or the condition for preserving the order in which the curves (2.2) follow one another in subcritical and critical positions).

In the Lyapunov-critical case, *three singular points merge* at the origin of the x_1x_2 plane if the differences (2.4) have different signs. Thus, the condition for a merging bifurcation is $g_1 > 0$ (or the condition for a change in the order in which the curves (2.2) follow one another in subcritical and critical positions).

Using (2.1)–(2.3) and (1.3), we find that $F'''_{20} - F'''_{10} = 6a_{12}^{-4}a_{21}^{-1}\Delta$. Hence it follows from (1.4) and (1.5) that

$$gg_1 = -6\Delta^2 a_{12}^{-4} a_{11}^{-3} a_{21}^{-1} \lambda_2^{-3} \left(\gamma_1\gamma_2 q a_{11}^{-1} a_{21}^{-1}\right)^* \tag{2.5}$$

Under our assumptions $\lambda_2 < 0$, $a^* > 0$, $(\gamma_1\gamma_2)^* > 0$, so that the sign of the right-hand side of (2.5) is defined by that of the quantity $a_{11}a_{11}^*a_{21}a_{21}^*$. The latter is positive, since a small neighbourhood of the critical position exists in which a_{11}^* and a_{21}^* have the same signs as in the critical position (by continuity).

Thus, $g_{21} > 0$. Consequently, the case of asymptotic stability ($g < 0$) of the point $x_1 = 0, x_2 = 0$ is uniquely associated with a generation bifurcation of singular points ($g_1 < 0$), and the case of instability ($g > 0$) with a merging bifurcation ($g_1 < 0$). Thus, a change in the stability of the symmetric solution is associated with the realization of a threefold singular point in the manifold of steady states—a cusp singularity. The bifurcation set of the cusp in a small neighbourhood of the threefold point is described by a semi-cubic parabola [8, 9]. The parameter values at which the cusp is replaced by its dual must be identical with those at which the curves (2.2) have a five-point contact at the origin, i.e. $F_{10}''' = F_{20}'''$ (a change of stability of a threefold singular point by realization of a butterfly singularity).

3. TWO-UNIT SYSTEMS WITH ROLLING: REDUCTION OF THE THREE-DIMENSIONAL PROBLEM TO TWO DIMENSIONS

Let us consider a mechanical system consisting of a biaxial driving unit and a uniaxial driven unit linked to the other unit by a hinge (Fig. 1). Let υ and u be the projections of the velocity of the mass centre C of the driving unit on its longitudinal and transverse axes, respectively, let $\omega = \vartheta^{\cdot}$ be the angular velocity of yaw of the driving unit, and let φ be the accumulation angle of the driven unit. Using the quasi-velocities υ and u, the real velocity ω and the variable φ instead of the holonomic coordinates x, y, ϑ and φ, one can split the initial system of differential equations for the plane-parallel motion of the two-unit system, which is of order eight, into two successively integrable subsystems, one of order five

$$\Phi_i\left(\upsilon^{\cdot}, u^{\cdot}, \omega^{\cdot}, \varphi^{\cdot\cdot}, \varphi^{\cdot}, \upsilon, u, \omega, \varphi\right) = 0 \quad (i = 1, \ldots, 4) \tag{3.1}$$

and the other of order three

$$\vartheta^{\cdot} = \omega, \quad x^{\cdot} = \upsilon\cos\vartheta - u\sin\vartheta, \quad y^{\cdot} = \upsilon\sin\vartheta + u\cos\vartheta \tag{3.2}$$

For the form of the functions Φ_i see [2].

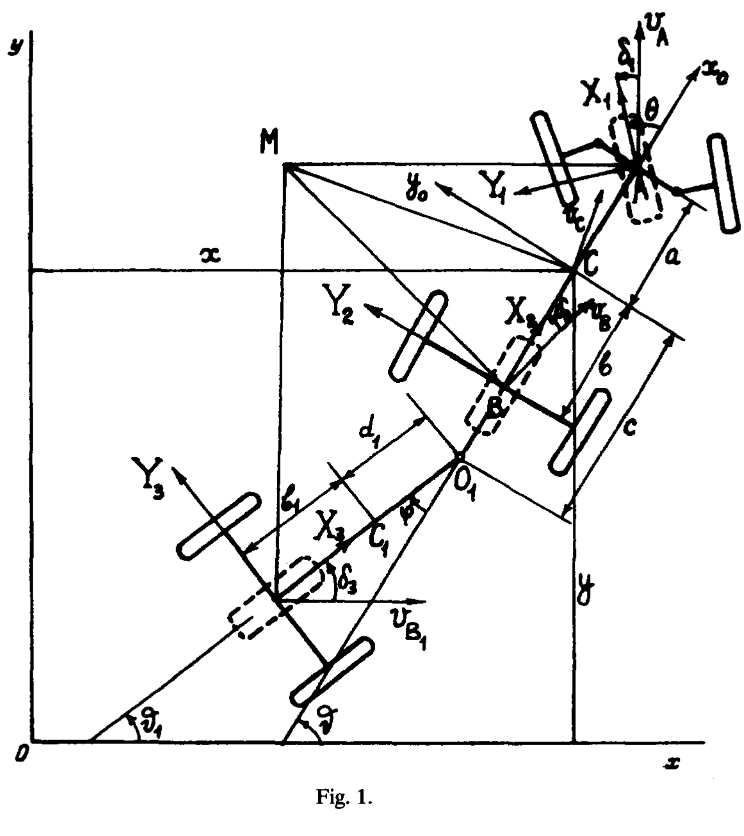

Fig. 1.

Let us consider motion at a constant velocity υ. It follows from (3.2) that steady solutions u = const, ω = const, φ = const of system (3.1) are represented in the reference plane (the xy plane) by circular trajectories of finite or infinite radius. It has been shown [2] that, at small values of θ, the problem of finding the singular points of system (3.1) may be reduced to the equations

$$-m\upsilon\omega + Y_1 + Y_2 + Y' = 0, \quad aY_1 - bY_2 - cY' = 0 \tag{3.3}$$
$$-m_1\omega[\upsilon\cos\varphi - (u - c\omega)\sin\varphi] + Y_3 = 0$$

where

$$Y' = m_1[-\upsilon\sin\varphi + \omega d_1 - (u - c\omega)\cos\varphi]\omega\sin\varphi \tag{3.4}$$

The unknown quantities in (3.3) are ω, u and φ. The Y_i are known functions of the angles of side slip δ_i, described in [2]. At $\upsilon = \upsilon_+$ the linearization matrix of system (3.1) in the neighbourhood of rectilinear motion of the system has one zero eigenvalue, while the other three eigenvalues have negative real parts.

Let us expand the left-hand sides of Eqs (3.3) in Taylor series in the neighbourhood of the point $\omega = 0$, $\upsilon = 0$, $\varphi = 0$. The linear approximations of the first two equations of system (3.3) with respect to ω and u are linearly dependent at the critical value υ_* of υ (the variable φ occurs in them non-linearly). Expressing φ via the third equation as a series in powers of ω, u and substituting into the first two equations, we obtain a system of two equations for steady motion of the initial system, accurate to within a given accuracy. The calculations will be carried out, dropping terms of order higher than three.

Since

$$\delta_3 = -\varphi + [\omega(c + d_1) - u]\upsilon^{-1} + \ldots$$

it follows from the third equation of (3.3) that

$$\varphi = -u\upsilon^{-1} + \left[(c + d_1)\upsilon^{-1} - m_1\upsilon k_3^{-1}\right]\omega + \ldots$$

For steady motion of the two-unit system

$$u = (b - ma\upsilon^2 k_2^{-1} l^{-1})\,\omega + \ldots$$

Therefore

$$\varphi = \left[c + d_1 - b + \upsilon^2\left(mak_2^{-1}l^{-1} - m_1 k_3^{-1}\right)\right]\upsilon^{-1}\omega + \ldots \tag{3.5}$$

It follows from (3.4) that

$$Y' = m_1^2\upsilon^2 k_3^{-1}\left\{-u\upsilon^{-1} + \left[(c + d_1)\upsilon^{-1} - m_1\upsilon k_3^{-1}\right]\omega\right\}\omega^2 + \ldots \tag{3.6}$$

This expression enables us to treat the first two equations of system (3.3) as independent, i.e. to reduce the three-dimensional problem of steady states to a two-dimensional problem. Changing from the variables ω and u to the variables δ_1 and δ_2 using formulae (1.1) of [2] and putting

$$\alpha = m_1\upsilon^4\left(m\varkappa_3 g^2 l^3\right)^{-1}\left[\upsilon^2(\varkappa_3 - \varkappa_2)(\varkappa_2\varkappa_3 g)^{-1} + c + d_1 - b)\right]$$
$$Y_i = k_i\delta_i - k_i'\delta_i^3 + \ldots, \quad k_i' = k_i^3(2G_i^2\varphi_i^2)^{-1}$$
$$\varkappa_i = k_i G_i^{-1}, \quad \varkappa_i' = k_i' \ G_i^{-1}, \quad G_1 = mgb_0$$
$$G_2 = mga_0, \quad G_3 = m_1 g, \quad a_0 = al^{-1}$$
$$b_0 = bl^{-1}, \quad Y_*' = Y'(mg)^{-1}$$

we can write these equations as

$$-\upsilon g^{-1}\omega+\varkappa_1 b_0\delta_1+\varkappa_2 a_0\delta_2-\varkappa_1' b_0\delta_1^3-\varkappa_2' a_0\delta_2^3+Y_*'+\ldots=0 \tag{3.7}$$

$$\varkappa_1\delta_1-\varkappa_2\delta_2-\varkappa_1'\delta_1^3+\varkappa_2'\delta_2^3-ca_0^{-1}b^{-1}Y_*'+\ldots=0$$

Here

$$Y_*'=\alpha l^3\upsilon^{-3}\omega^3+\ldots,\quad \omega=\upsilon l^{-1}\left[\theta-\delta_1+\tfrac{1}{3}(\theta-\delta_1)^3+\delta_2+\tfrac{1}{3}\delta_2^3+\ldots\right]$$

4. THE STABILITY OF RECTILINEAR MOTION (θ = 0) OF THE TWO-UNIT SYSTEM AT CRITICAL VELOCITIES

The critical velocity is defined by formula (1.15) in [2], which may be transformed as follows:

$$\upsilon_+^2=\varkappa_1\varkappa_2 gl(\varkappa_1-\varkappa_2)^{-1},\quad \varkappa_1=k_1/G_1$$

$$\varkappa_2=k_2/G_2,\quad G_1=mgbl^{-1}-m_1gb_1(c-b)l^{-1}L_1^{-1}$$

$$G_2=mgal^{-1}+m_1gb_1(a+c)l^{-1}L_1^{-1}$$

where k_i are the cornering stiffnesses. In this case b_1 = 0. Since

$$\gamma_1=-\frac{\varkappa_1 bg+\upsilon^2}{\varkappa_2 ag-\upsilon^2},\quad \frac{d\gamma_1}{d\upsilon}<0$$

$$\gamma_1\big|_{\upsilon=\upsilon_+}=\gamma_2=\frac{\varkappa_1}{\varkappa_2}$$

it follows that at $\upsilon=\upsilon_+$ we have $(\gamma_1-\gamma_2)^*>0$. The condition $F_{20}'''-F_{10}'''<0$ is satisfied if and only if

$$d_1>d_1^*,\quad d_1^*=b-c+m\varkappa_3\upsilon_+^4(2m_1\varkappa_1\varkappa_2g^2l)^{-1}\left(\varkappa_1\varphi_2^{-2}-\right. \tag{4.1}$$

$$\left.-\varkappa_2\varphi_1^{-2}\right)\left[1+c(\varkappa_1b+\varkappa_2a)(\varkappa_1-\varkappa_2)^{-1}(ab)^{-1}\right]^{-1}-\upsilon_+^2(\varkappa_3-\varkappa_2)(\varkappa_2\varkappa_3g)^{-1}+$$

$$+m\varkappa_3\upsilon_+^2\left(\varkappa_1^2+\varkappa_1\varkappa_2+\varkappa_2^2\right)\left\{3m_1g\varkappa_1^2\varkappa_2^2\left[1+c(\varkappa_1b+\varkappa_2a)(\varkappa_1-\varkappa_2)^{-1}a^{-1}b^{-1}\right]\right\}^{-1}$$

The last term in (4.1) is generated by the cubic terms in ω and is substantially less than the preceding terms; hence any quantitative corrections that it induces may be ignored. By previous reasoning (Section 2), there is a generation bifurcation at the origin of the phase space, while the boundary $\upsilon=\upsilon_+$ of the stability domain in the parameter space of the two-unit system is safe (in the sense of [10]). When $d_1<d_1^*$ one has a merging bifurcation, and the hyperplane $\upsilon=\upsilon_+$ is unsafe.

5. BIFURCATION SETS OF STEADY STATES OF THE TWO-UNIT SYSTEM ($\theta\neq 0$)

Putting $Y_i^*=Y_iG_i^{-1}$ and using (3.6), we write the first two equations of (3.3) in the form

$$-\upsilon^2(gl)^{-1}(\theta-\delta_1+\delta_2)+b_0Y_1^*+a_0Y_2^*+\alpha(\theta-\delta_1+\delta_2)^3+\ldots=0 \tag{5.1}$$

$$Y_1^*-Y_2^*-ca_0^{-1}b^{-1}\alpha(\theta-\delta_1+\delta_2)^3+\ldots=0$$

Define a new variable by

$$Y=\upsilon^2c_1(bg)^{-1}\delta_1+c_2Y_1^*(\delta_1)=\upsilon^2c_1(bg)^{-1}(\theta+\delta_2)+c_3Y_2^*(\delta_2)$$

$$(c_1=ca^{-1},\quad c_2=1+c_1,\quad c_3=1-cb^{-1})$$

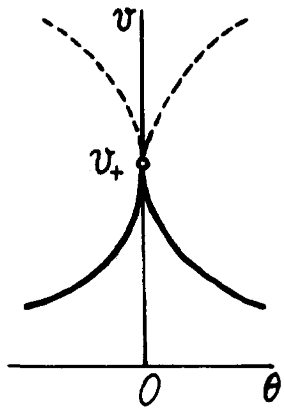

Fig. 2.

Expressing δ_1 and δ_2 in terms of Y and retaining terms of order up to three, we obtain a cubic equation in Y. Equating the discriminant of the equation to zero we find the bifurcation set of the two-parameter (θ, υ)-family of steady states

$$\theta^2 = \gamma w^3 + o(w^3), \quad w = \upsilon_+^2 \upsilon_0^{-2} - 1 \tag{5.2}$$

$$\gamma = \tfrac{4}{27} c_3^2 \left(1 - c_1 \upsilon_+^2 \beta_3 g^{-1} b^{-1}\right)^{-2} \alpha_1^{-3} \beta^{-1}$$

$$\beta_3 = \left(\upsilon_+^2 c_1 b^{-1} g^{-1} + \varkappa_2 c_3\right)^{-1}, \quad \alpha_1 = \upsilon_+^4 c_1^2 b^{-2} g^{-2} + \upsilon_+^2 c_1 b^{-1} g^{-1} (\varkappa_1 c_2 + \varkappa_2 c_3) +$$

$$+ \varkappa_1 \varkappa_2 c_2 c_3, \quad \beta = (\varkappa_1 - \varkappa_2) a^4 b^4 \varkappa_1^{-1} \varkappa_2^{-1} [\varkappa_1 b (a + c) - \varkappa_2 a (b - c)]^{-4} \{ \tfrac{1}{2} (\varkappa_1 \varphi_2^{-2} - \varkappa_2 \varphi_1^{-2}) -$$

$$- \alpha (\varkappa_1 - \varkappa_2)^4 \varkappa_1^{-3} \varkappa_2^{-3} \left[1 + c_1 b^{-1} (\varkappa_1 b + \varkappa_2 a)(\varkappa_1 - \varkappa_2)^{-1}\right] \}$$

Here υ_0 is the critical value of the velocity of angular motion of the two-unit system for the given values of θ. The sign of γ depends on the value of d_1: if $d_1 \lesseqgtr d_1^*$, then $\gamma \gtreqless 0$. Figure 2 shows the curve (5.2) in the neighbourhood of the point $(0, \upsilon_+)$: the solid curve for $d_1 < d_1^*$ and the dashed curve for $d_1 > d_1^*$. As d_1 increases and passes through d_1^* the cusp is reorganized into its dual (transition to a safe part of the boundary of the stability domain). The bifurcation set represented by the solid curve has been described previously [6, 11].

6. COMPUTER SIMULATION

For the numerical determination of the set of parameters of the two-unit system at which the number of singular points changes, we supplement the three equations of the steady states by a fourth, obtained by equating the Jacobian of the system to zero. The unknowns are ω, u, φ, θ and υ. The manifold of steady states is found by the method of continuation [12] as a function of the two parameters θ and υ, having taken $\omega = 0$, $u = 0$, $\varphi = 0$, $\theta = 0$, $\upsilon = \upsilon_+$ as the starting point. We take $m = 5310$ kg, $m_1 = 6481$ kg, $a = 1.92$ m, $b = 0.82$ m, $c = b$, $d_1 = 14$ m, $k_1 = 305{,}091$ N, $k_2 = 103{,}496$ N, $k_3 = 154{,}079$ N, and $\varphi_1 = \varphi_2 = 0.8$. Then $\upsilon_+ = 9.436$ m/s and $d_1^* = 9.509$ m. The curve in the upper part of Fig. 3 is the section of the bifurcation surface by a plane $d_2 =$ const, where $d_1 = 9.3$ m. On the left we show a fragment of the bifurcation set in a small neighbourhood of the point $\theta = 0$, $\upsilon = \upsilon_+$, confirming the fact that the boundary of the stability domain $\upsilon = \upsilon_+$ is unsafe at $d_1 < d_1^*$. On the right is the same for $d_1 = 9.7$ m; in this case the boundary is safe (the sections of the bifurcation surface at $d_1 = 9.3$ m and $d_1 = 9.7$ m are practically the same on the scale chosen here). At $d_1 = d_1^*$ a "butterfly" singularity is obtained at the point $(0, \upsilon_+)$.

Figure 4 ($k_3 = 254{,}079$ N, $d_1 = 13.56$ m $< d_1^*$) shows that in a small neighbourhood of the point $(0, \upsilon_+)$ four other cusps exist. The symbol $D(s)$ denotes domains with s steady states. The qualitative structure of the bifurcation curves is the same as that for $k_3 = 154{,}079$ N and $d_1 = 9.3$ m. The difference is that in Fig. 3 the upper cusps are farther apart. The domain in Fig. 4 is formed by the intersection of the cusp and two symmetrically placed swallowtails. Because of the evolution of these swallowtails (birth and death) as the parameter d_1 varies, the unsafe-safe property of the boundary $\upsilon = \upsilon_+$ of the stability domain changes at d_1^*.

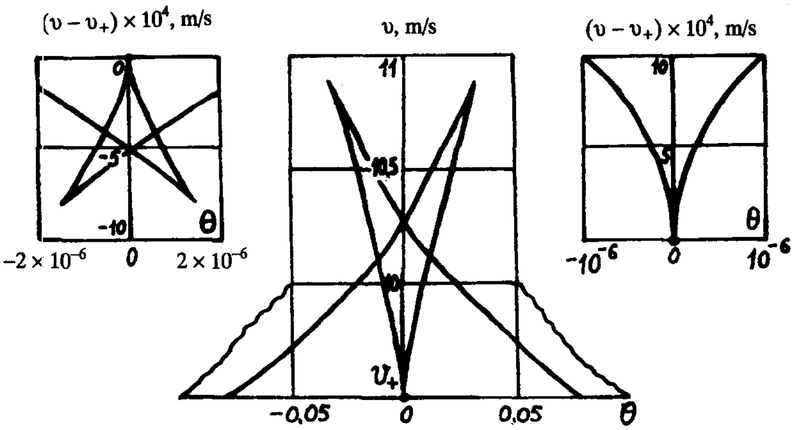

Fig. 3.

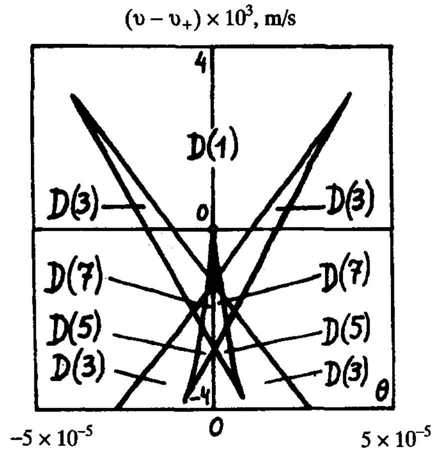

Fig. 4.

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